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ADDENDUM

1/N expansion for the Yukawa potential revisited: II

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Abstract. Using the form prescribed by Moreno and Zepeda for the N-dimensional Yukawa potential and employing the method of $1/N$ expansion we have obtained in a recent paper the energies of the ground state and the first excited state of a three-dimensional Yukawa potential. In the present addendum we introduce a modified prescription for the N-dimensional Yukawa potential which enables us also to calculate the energies of the higher angular momentum states. Our results are in very good agreement with the accurate numerical values.

It is well known that the Yukawa potential plays an important role in various branches of physics. Unfortunately, this potential does not admit an exact solution and therefore various approximate analytic and numerical methods (Hulthen and Laurikainen 1951, Rogers *et al* 1970, McEnnen *et al* 1976, Gazeau and Maquet 1978, Grant and Lai 1979, Dutta *et al* 1985 and references therein) have been employed over the past several years to obtain its energy spectrum. Recently, Moreno and Zepeda (1984) have applied the $1/N$ expansion technique to solve the Schrödinger equation with a Yukawa potential, $V(r) = -(a/r)\,e^{-br}$ and have proposed an approximate analytic formula for the ground state energy which is in very good agreement with the numerical results. They have also shown that the $1/N$ expansion for the ground state energy is not convergent when β $(=9b/4a)$ approaches 1. This leads to an intriguing controversy over the very applicability of the method and it is this aspect of the problem that has motivated our recent investigation (Chatterjee 1985, hereafter referred to as I). In I, we have calculated the energies of the ground state and the first excited state of a particle bound to a Yukawa potential following the method of Mlodinow and Shatz (1982). By collecting terms of the same order in β we have shown that the $1/N$ expanded energy can in fact, be reckoned as a series in powers of β, whereupon the awkward divergence problem disappears altogether. We have furthermore shown that the results provided by the method of $1/N$ expansion is identical to those obtained by using the hypervirial equations with the Hellman-Feynman theorem (Grant and Lai 1979) and the analytic perturbation theory (McEnnen *et al* 1976). But the form of the N-dimensional Yukawa potential as proposed by Moreno and Zepeda and followed in I could be used only for s states. In the present addendum we give a modified prescription for the Yukawa potential to obtain energies for the higher angular momentum states also.

The N-dimensional Schrödinger equation in spherical polar coordinates (in units $m = \hbar = 1$) is given by

$$\left[-\frac{1}{2}\left(\frac{\partial^2}{\partial r^2}+\frac{N-1}{r}\frac{\partial}{\partial r}\right)+\frac{L^2}{2r^2}+V(r)\right]\Psi(\boldsymbol{r}) = E\Psi(\boldsymbol{r}) \tag{1}$$

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where L is the angular momentum operator in N dimensions. Now substituting

$$\Psi(r) = r^{-(N-1)/2} u(r) Y_{lm}(\Omega) \tag{2}$$

where $Y_{lm}(\Omega)$ is an eigenfunction of L^2 belonging to the eigenvalue $(l+N-2)$ and using the following prescription for the Yukawa potential:

$$V(r) = -\frac{a}{r}\exp\left(-\frac{(3+2l)^2}{k^2}br\right) \tag{3}$$

we obtain from (1)

$$-\frac{1}{2}\frac{d^2}{dr^2}u(r) + k^2\left[\frac{(1-1/k)(1-3/k)}{8r^2} - \frac{\tilde{a}}{r}\exp\left(-\frac{(3+2l)^2}{k^2}br\right)\right]u(r) = Eu(r) \tag{4}$$

where $k = N+2l$, $\tilde{a} = a/k^2$. The leading-order energy eigenvalue is given by

$$E_\infty = k^2 E^{(-2)} = k^2\left(\frac{1}{8r_0^2} - \frac{\tilde{a}}{r_0}\right) \tag{5}$$

where r_0 is obtained by minimising E_∞.

Let us now define $x = r - r_0$ and substitute

$$u(r) = e^{\Phi(x)} \tag{6}$$

in (4) which then reduces to a Riccati equation. Next, expanding the energy E and the first derivative of the function $\Phi(x)$ in powers of $1/k$ and then equating the terms of the same order in k, we end up with a set of recurrence relations which can be solved to yield the higher-order corrections. The energy and the radial wavefunction corresponding to the choice (6) are then given by

$$\frac{E}{a^2} = -\frac{2}{(k-1)^2} + g - \tfrac{1}{8}(k^2-k)g^2 + \tfrac{1}{96}(k^4-k^3-k^2+k)g^3 - \tfrac{1}{384}(k^6 - \tfrac{9}{4}k^5 + \tfrac{1}{2}k^4 + 2k^3 - \tfrac{3}{2}k^2 + \tfrac{1}{4}k)g^4 + \ldots \tag{7}$$

$$R(r) \sim r^{(k-N)/2}\exp\left(-\frac{2a}{k-1}r + \frac{(k-1)g^2a^2}{8}r^2 - \frac{(k^3-k^2+k-1)g^3a^2}{96} \times r^2 - \frac{(k-1)g^3a^3}{36}r^3 + \ldots\right) \tag{8}$$

where

$$g = \frac{(3+2l)^2}{k^2}\frac{b}{a}.$$

For $N=3$ which is the dimension of interest, the screening parameter g is given by $g = b/a$. In the Coulomb limit ($b=0$) we get

$$E = -\frac{2a^2}{(k-1)^2} \tag{9}$$

$$R(r) \sim r^{(k-N)/2}\exp\left(-\frac{2a}{k-1}r\right). \tag{10}$$

It is interesting to note that the expressions (9) and (10) reproduce the exact energies and radial wavefunctions for the three-dimensional Coulomb potential for the states

that are nodeless in the region $0<r<\infty$ and that behave as r^l at $r=0$. These states are given by $n-l-1=0$. Since the eigenfunctions of a Yukawa potential may be thought of as perturbed Coulomb wavefunctions, we expect that the number of nodes associated with the radial wavefunctions of a Yukawa potential will again be determined by $(n-l-1)$. Therefore the energy eigenvalues for the states 1s, 2p, 3d, etc, may be obtained from (7). The results are

$$E_{1s}=-a^2(\tfrac{1}{2}-g+\tfrac{3}{4}g^2-\tfrac{1}{2}g^3+\tfrac{11}{16}g^4+\ldots) \tag{11a}$$

$$E_{2p}=-a^2(\tfrac{1}{8}-g+\tfrac{5}{2}g^2-5g^3+\tfrac{95}{4}g^4+\ldots) \tag{11b}$$

$$E_{3d}=-a^2(\tfrac{1}{18}-g+\tfrac{21}{4}g^2-21g^3+\tfrac{1701}{8}g^4+\ldots). \tag{11c}$$

Next we choose $u(r)=(x-\Sigma_{n=1}^{\infty}C^{(n)}k^{-n})\exp(\Phi(x))$ which corresponds to the states for which the $n-l-1=1$. In general, the choice $u(r)=\Pi_{m=1}^{n-1}(x-\Sigma_{j=1}^{\infty}C_m^{(j)}k^{-j})\exp(\Phi(x))$ corresponds to ns, $(n+1)$p, $(n+2)$d, $(n+3)$f, etc. For 2s and 3p, we obtain

$$E_{2s}=-a^2(\tfrac{1}{8}-g+3g^2-7g^3+\ldots) \tag{12a}$$

$$E_{3p}=-a^2(\tfrac{1}{18}-g+\tfrac{25}{4}g^2-30g^3+\ldots). \tag{12b}$$

From (11) and (12) we now calculate the energies of the states 1s, 2s, 2p, 3p and 3d (in atomic units) for different values of the screening parameter g (see table 1). Our results are in fair agreement with the accurate numerical values obtained by Rogers *et al* (1970). It is also clear that, for a given n, the states with higher l will, in general, have higher energy. However, in the Coulomb limit the accidental l degeneracy is restored.

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Table 1. Energy eigenvalues (in atomic units). First line, $1/N$ expansion; second line, accurate numerical results of Rogers *et al* (1970).

g	1s	2s	2p	3p	3d
0	−1.000 0	−0.250 0	−0.250 0	−0.111 1	−0.111 1
(Coulomb)	−1.000 0	−0.250 0	−0.250 0	−0.111 1	−0.111 1
0.002	−0.996 01	−0.246 02	−0.246 02	−0.107 16	−0.107 15
	−0.996 0	−0.246 0	−0.246 0	−0.107 2	−0.107 2
0.005	−0.990 04	−0.240 15	−0.240 12	−0.101 42	−0.101 37
	−0.990 0	−0.240 1	−0.240 1	−0.101 4	−0.101 4
0.01	−0.980 15	−0.230 58	−0.230 49	−0.092 36	−0.092 12
	−0.980 1	−0.230 6	−0.230 5	−0.092 31	−0.092 12
0.02	−0.960 59	−0.212 29	−0.211 93	−0.075 63	−0.075 04
	−0.960 6	−0.212 3	−0.211 9	−0.075 70	−0.075 03
0.025	−0.95092	−0.203 52	−0.202 99	−0.067 99	−0.067 18
	−0.950 9	−0.203 6	−0.203 0	−0.068 16	−0.067 15
0.05	−0.903 63	−0.063 25	−0.161 55	−0.034 86	−0.347 77
	−0.903 6	−0.163 5	−0.161 5	−0.037 12	−0.033 83

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